سلم تصحيح الرياضيات (2)

السنة الثانية / إنتاج

الفصل الثاني 2023 - 2024

السؤال الأول: (35) درجة

*) $I_1 = \int \frac{x^2}{1+x^6}\,dx$

$= \int \frac{x^2}{1+(x^3)^2}\,dx$ (6)

نفرض $t = x^3 \Rightarrow dt = 3x^2\,dx$

$\Rightarrow x^2\,dx = \frac{1}{3}\,dt$

$\Rightarrow I_1 = \frac{1}{3}\int \frac{dt}{1+t^2}$

$= \frac{1}{3}\arctan(t) + c$

$= \frac{1}{3}\arctan(x^3) + c$

*) $I_2 = \int \frac{1}{x^2\sqrt{4-x^2}}\,dx$ (6)

نفرض $x = 2\sin t \Rightarrow dx = 2\cos(t)\,dt$

$t = \arcsin\left(\frac{x}{2}\right)$

$\Rightarrow I_2 = \int \frac{1}{4\sin^2 t\sqrt{4-4\sin^2 t}}\cdot 2\cos(t)\,dt$

$= \frac{1}{4}\int \frac{1}{\sin^2(t)}\,dt$

$= -\frac{1}{4}\cot(t) + c$

$= -\frac{1}{4}\cot\left(\arcsin\left(\frac{x}{2}\right)\right) + c$

2) $I_3 = \int \arctan(x)\,dx$

نكامل بالتجزئة

$u = \arctan(x) \Rightarrow du = \frac{1}{1+x^2}\,dx$

$dv = dx \Rightarrow v = x$ (5)

$\Rightarrow I_3 = u.v - \int v\,du$

$= x\arctan(x) - \int \frac{x}{1+x^2}\,dx \cdot \frac{2}{2}$

$= x\arctan(x) - \frac{1}{2}\ln(1+x^2) + c$

*) $I_4 = \int \frac{1}{\sin(x)}\,dx$

نفرض $t = \tan\left(\frac{x}{2}\right) \Rightarrow$ (6)

$x = 2\arctan(t) \Rightarrow$

$dx = \frac{2\,dt}{1+t^2}$, $\sin(x) = \frac{2t}{1+t^2}$

$\int \frac{dx}{\sin(x)} = \int \frac{\frac{2\,dt}{1+t^2}}{\frac{2t}{1+t^2}}$

$= \int \frac{dt}{t} = \ln|t| + c$

$= \ln\left|\tan\left(\frac{x}{2}\right)\right| + c$

نفرض $t = x + \frac{1}{2} \Rightarrow$

$dt = dx$

$$I_6 = \int \frac{1}{t^2 + \frac{3}{4}}\, dt$$

$$= \frac{1}{\frac{\sqrt{3}}{2}} \arctan\left(\frac{t}{\frac{\sqrt{3}}{2}}\right) + c$$

$$= \frac{2}{\sqrt{3}} \arctan\left(\frac{2t}{\sqrt{3}}\right) + c$$

$$= \frac{2}{\sqrt{3}} \arctan\left(\frac{2.(x+\frac{1}{2})}{\sqrt{3}}\right) + c$$

$$= \frac{2}{\sqrt{3}} \arctan\left(\frac{2x+1}{\sqrt{3}}\right) + c$$

السؤال الثاني: (١٥ درجة)

$$I = \int \frac{x^2 - 2}{(x^2-1)(x-1)}\, dx$$

$$= \int \frac{x^2-2}{(x-1)(x+1)(x-1)}\, dx$$

$$= \int \frac{x^2-2}{(x+1)(x-1)^2}\, dx$$ (3)

نجد A, B, C ثوابت حقيقية بحيث

$$\frac{x^2-2}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$\Rightarrow x^2 - 2 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$$

*) $$I_5 = \int \frac{1+\sqrt[4]{x}}{x+\sqrt{x}}\, dx$$

نفرض $x = t^4 \Rightarrow dx = 4t^3\,dt$

$t = \sqrt[4]{x} = x^{\frac{1}{4}}$

$\sqrt{x} = x^{\frac{1}{2}} = (t^4)^{\frac{1}{2}} = t^2$ (6)

$\sqrt[4]{x} = x^{\frac{1}{4}} = (t^4)^{\frac{1}{4}} = t$

$$\int \frac{1+\sqrt[4]{x}}{x+\sqrt{x}}\, dx = \int \frac{1+t}{t^4+t^2} \cdot 4t^3\, dt$$

$$= 4\int \frac{t^2+t}{t^2+1}\, dt$$

$$= 4\int \left(1 + \frac{t-1}{t^2+1}\right) dt$$

$$= 4\int \left(1 + \frac{t}{t^2+1} - \frac{1}{t^2+1}\right) dt$$

$$= 4\left[t + \frac{1}{2}\ln|t^2+1| - \arctan(t)\right]$$

$$= 4\left[\sqrt[4]{x} + \frac{1}{2}\ln|\sqrt{x}+1| - \arctan(\sqrt[4]{x})\right]$$

*) $$I_6 = \int \frac{1}{x^2+x+1}\, dx$$ (6)

$$= \int \frac{1}{x^2+x+\frac{1}{4}-\frac{1}{4}+1}\, dx$$

$$= \int \frac{1}{(x+\frac{1}{2})^2 + \frac{3}{4}}\, dx$$

(2)

$$= \lim_{d \to -\infty} \left[\arctan(x)\right]_d^0$$

(4)

$$= \lim_{d \to -\infty} \left[-\arctan(d)\right]$$

$$= -\arctan(-\infty) = \frac{\pi}{2}$$ متقارب

$$M_2 = \int_0^{\infty} \frac{dx}{1+x^2}$$

$$= \lim_{b \to \infty} \int_0^b \frac{dx}{1+x^2}$$

$$= \lim_{b \to \infty} \left[\arctan(x)\right]_0^b$$

(4)

$$= \lim_{b \to \infty} \left[\arctan(b)\right] =$$

$$= \arctan(\infty) = \frac{\pi}{2}$$ متقارب

$$\Rightarrow I_1 = M_1 + M_2 = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

ومنه التكامل I_1 متقارب.

~~*)~~ $$\cancel{I_2 = \int_0^1 \frac{dx}{\sqrt{1-x}}}$$

معتل في $x=1$

$$I_2 = \int_0^1 \frac{dx}{\sqrt{1-x}} + \cancel{\int_1^{\infty} \frac{dx}{\sqrt{1-x}}}$$

M_1 M_2

$$x=-1 \Rightarrow -1 = 4A \Rightarrow$$

$$x=1 \Rightarrow -1 = 2C \Rightarrow C = -\frac{1}{2}$$

(3)

$$x=0 \Rightarrow -2 = A - B + C$$

$$-2 = -\frac{1}{4} - B - \frac{1}{2} \Rightarrow$$

$$B = \frac{5}{4}$$

$$I = \int \left(\frac{-\frac{1}{4}}{x+1} + \frac{\frac{5}{4}}{x-1} + \frac{-\frac{1}{2}}{(x-1)^2}\right) dx$$

$$= -\frac{1}{4} \ln|x+1| + \frac{5}{4} \ln|x-1| - \frac{1}{2} \frac{(x-1)^{-1}}{-1} + C$$

(4)

$$= -\frac{1}{4} \ln|x+1| + \frac{5}{4} \ln|x-1| + \frac{1}{2(x-1)} + C$$

السؤال الثالث: (12 درجة)

$$*) I_1 = \int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$$

التكامل معتل في $+\infty$ و $-\infty$

$$I_1 = \underbrace{\int_{-\infty}^0 \frac{dx}{1+x^2}}_{M_1} + \underbrace{\int_0^{\infty} \frac{dx}{1+x^2}}_{M_2}$$

$$M_1 = \int_{-\infty}^0 \frac{dx}{1+x^2} = \lim_{a \to -\infty} \int_a^0 \frac{dx}{1+x^2}$$

(3)

$$= \frac{r^2}{2}\left[t + \frac{1}{2}\sin(2t)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{r^2}{2}\left[\frac{\pi}{2} - 0\right] = \frac{\pi r^2}{4} \quad (3)$$

$$\Rightarrow S = 4S_1 = 4 \cdot \frac{\pi r^2}{4} = \pi r^2$$

ليكن L هو محيط الدائرة عندئذ

$$L = 4L_1$$

$$L_1 = \int_0^r \sqrt{1 + y'^2}\, dx \quad (3)$$

$$\sqrt{1+y'^2} = \frac{r}{\sqrt{r^2 - x^2}}$$

$$L_1 = \int_0^r \frac{r}{\sqrt{r^2 - x^2}}\, dx$$

$$= r\left[\arcsin\left(\frac{x}{r}\right)\right]_0^r \quad (3)$$

$$= r\left[\arcsin(1) - \arcsin(0)\right] = r\frac{\pi}{2}$$

$$\Rightarrow L = 4L_1 = 4 \cdot r \cdot \frac{\pi}{2} = 2\pi r$$

ثانياً :

$$x = r\cos(t), \quad y = r\sin(t) \quad (2)$$
$$0 \le t \le \pi$$

$$S_x = 2\pi \int_0^{\pi} y(t) \cdot \sqrt{x_t'^2 + y_t'^2}\, dt \quad (3)$$

$$x_t'^2 + y_t'^2 = r^2\sin^2(t) + r^2\cos^2(t)$$
$$= r^2\left(\sin^2(t) + \cos^2(t)\right)$$
$$= r^2(1) = r^2$$

*) $I_2 = \int_0^1 \frac{dx}{\sqrt{1-x}}$

معتل عند $x = 1$

$$I_2 = \lim_{k \to 1^-} \int_0^k \frac{dx}{\sqrt{1-x}}$$

$$= \lim_{k \to 1^-} -2\left[\sqrt{1-x}\right]_0^k \quad (4)$$

$$= \lim_{k \to 1^-} -2\left[\sqrt{1-k} - 1\right]$$

$$= -2\left[\sqrt{1-1} - 1\right] = -2(-1) = 2$$

ومنه التكامل متقارب

السؤال الرابع : (28 درجة)

أولاً :

$$x^2 + y^2 = r^2 \Rightarrow \quad (3)$$

$$y^2 = r^2 - x^2 \Rightarrow y = \sqrt{r^2 - x^2}$$

لتكن S مساحة الدائرة عندئذ

$$S = 4S_1$$

$$S_1 = \int_0^r y\, dx = \int_0^r \sqrt{r^2 - x^2}\, dx \quad (3)$$

نعوض $x = r\sin t \Rightarrow dx = r\cos(t)\, dt$

$$x = 0 \Rightarrow t = 0$$
$$x = r \Rightarrow t = \frac{\pi}{2} \quad (2)$$

$$S_1 = \int_0^{\frac{\pi}{2}} \sqrt{r^2 - r^2\sin^2 t} \cdot r\cos(t)\, dt$$

$$= r^2 \int_0^{\frac{\pi}{2}} \cos^2(t)\, dt = r^2 \int_0^{\frac{\pi}{2}} \frac{1 + \cos(2t)}{2}\, dt$$

(4)

$\sqrt{x_t'^2+y_t'^2} = \sqrt{r^2} = r$

$y \cdot \sqrt{x_t'^2+y_t'^2} = r \cdot \sin(t) \cdot r = r^2 \sin(t)$

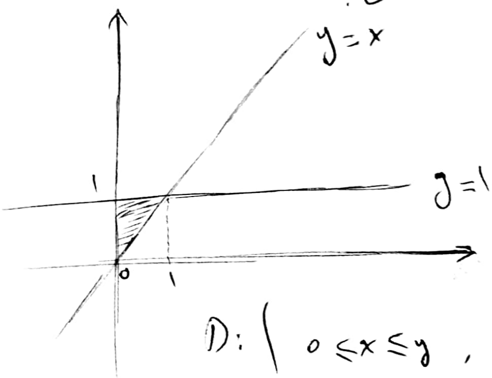
$S_x = 2\pi \int_0^{\pi} r^2 \cdot \sin(t)\, dt$

$= 2\pi r^2 \left[-\cos(t)\right]_0^{\pi}$ (3)

$= 2\pi r^2\left[-\cos(\pi) + \cos(0)\right]$

$= 2\pi r^2\left[1+1\right] = 4\pi r^2$

السؤال الخامس:



(4)

$D: \{ 0 \le x \le y \ , \ 0 \le y \le 1 \}$

$I = \iint_D y^2 e^{xy}\, dx\, dy$

$= \int_0^1 \int_0^y y^2 e^{xy}\, dx\, dy$ (4)

$= \int_0^1 \left[\int_0^y y^2 e^{xy}\, dx\right] dy$

$= \int_0^1 \left[y^2 \cdot \frac{1}{y} e^{xy}\right]_0^y dy$ (4)

$= \int_0^1 (y e^{y^2} - y)\, dy$ (3)

$= \left[\frac{1}{2} e^{y^2} - \frac{y^2}{2}\right]_0^1 = \frac{1}{2} e - 1$

انتهى السلم

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